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Formally Renormalizable Gravitationally Self-Interacting String Models

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It has recently been shown how the effect of the divergent part of the gravitational self-interaction for a classical string model in four dimensions can be allowed for by a renormalization of its stress-energy tensor and in the elastic case a corresponding renormalization of the off-shell action. It is shown here that that it is possible to construct a new category of elastic string models for which this effect is describable as a renormalization in the stricter "formal" sense, meaning that it only requires a rescaling of one of the fixed parameters characterizing the model.

1. INTRODUCTION

The gravitational self-interaction for a classical string model in a four-dimensional background has recently been shown [1] to have a divergent part that, when suitably regularized, can be effectively be absorbed by a renormalization of the surface stress-energy tensor of the worldsheet. In the particularly simple case of a Nambu–Goto model the renormalization is trivial in the sense that (contrary to what had been suggested by earlier work) the divergent part actually vanishes [2]. For more general elastic string models, the renormalization of the "on-shell" stress energy tensor has been shown [3] to correspond to a nontrivial renormalization of the relevant variational action, in which the appropriate adjustment has precisely the form that is obtained as the four-dimensional specialization of a very general formula [4] that has recently been obtained from a very different approach based on an analysis of the "off-shell" action in a background of arbitrary dimension.

The physical example that provided the original motivation for this line of investigation was the case of cosmic strings described by the "transonic"

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string model [5], which provides an effective large-scale description of the effect of short-wavelength "wiggles" in an underlying Nambu–Goto model. In this case the string model is qualitatively modified by the effect of the renormalization, in the sense that the resulting "dressed" model no longer has the special transonic character (and the ensuing integrability properties) of the original "bare" model. The purpose of the present article is to answer the question of whether there is a category of gavitationally self-interacting elastic string models that are renormalizable in the stricter "formal" sense, meaning that the algebraic form of the "bare" model is qualitatively preserved in the corresponding "dressed" model in the sense that the renormalization is describable just as a readjustment of the free parameters specifying the particular model within the category.

2. "BARE" STRING MODELS

As explained in more detail in refs. 3 and 4, we are concerned with models governed by an action $\mathcal{I}$ that is specified as an integral over the string worldsheet that is specifiable as the two-dimensional imbedding given by $x^\mu = \overline{x}^\mu\{\sigma\}$ in terms of intrinsic coordinates σ^i, ($i = 0, 1$), so that the induced surface metric will have the form $\gamma_{ij} = g_{\mu\nu}\, \overline{x}^\mu_{,i}\, \overline{x}^\nu_{,j}$, where $g_{\mu\nu}$ is the spacetime metric of the four-dimensional background with local coordinates x^μ. In terms of such a worldsheet, the action integral will have the form

$$\mathcal{I} = \int \overline{\mathcal{L}} \|\gamma\|^{1/2}\, d^2\sigma \tag{1}$$

where $|\gamma|$ is the determinant of the induced metric, and Λ is the relevant Lagrangian scalar. In the absence of gravitational interaction—other than what is automatically allowed for by the large-scale curvature of the background metric $g_{\mu\nu}$—the Lagrangian scalar for a model of the simple elastic kind under consideration here would be given by a master function Λ depending just on the gradient of a freely variable scalar stream function ψ on the worldsheet. However, to allow for the effect of shortwave gravitational perturbations $g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu}$ in terms of a linearized gravitational field tensor $h_{\mu\nu} = \delta g_{\mu\nu}$ it is evidently necessary to augment the Lagrangian by a corresponding gravitational coupling term so that it takes the form

$$\overline{\mathcal{L}} = \Lambda + \tfrac{1}{2}\overline{T}^{\mu\nu} h_{\mu\nu} \tag{2}$$

where $\overline{T}^{\mu\nu}$ is the relevant surface stress-energy tensor, which is given by $\overline{T}^{\mu\nu} = 2\|\gamma\|^{-1/2}\, \partial(\Lambda\|\gamma\|^{1/2})/\partial g_{\mu\nu}$. The corresponding dynamical equations are given by the requirement that the action integral should be invariant with respect to local variations of the worldsheet imbedding and of the internal field ψ.

3. "DRESSED" STRING MODELS

If the field $h_{\mu\nu}$ were due just to passing gravitational waves from an external source the model that has just been described would automatically be well defined and well behaved as it stands. Allowance for self-gravitation gives rise to difficulties of two kinds. The hardest part is the evaluation of the finite long-range contribution including backreaction from emitted radiation, which, except for very simple configurations will be tractable in practice only in a rather approximate manner. However, although it may be important in the long run, this finite contribution will usually have a negligible effect on the short-time-scale dynamics, for which the divergent short-range part of the self-interaction will dominate. The subject of the present discussion is this latter part, whose treatment requires the introduction of a regularization whereby it will be obtained [1] in the form

$$\hat{h}_{\mu\nu} = 2\mathrm{G}\,\hat{l}(2\overline{T}_{\mu\nu} - \overline{T}^{\sigma}_{\sigma}g_{\mu\nu}) \tag{3}$$

where, as usual for a string self-interaction in four dimensions, the proportionality factor has the form $\hat{l} = \ln\{\Delta^2/\delta_*^2\}$ in terms of an "ultraviolet" cutoff length scale δ_* representing the effective thickness of the string, and a much larger "infrared" cutoff Δ given by a length scale characterizing the large-scale geometry of the string configuration.

This provides a system [1] that will be describable in terms of the finite part $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \hat{h}_{\mu\nu}$ that is left over when the divergent part is subtracted out, by a renormalized Lagrangian of the form

$$\overline{\mathcal{L}} = \tilde{\Lambda} + \tfrac{1}{2}\overline{T}^{\mu\nu}\tilde{h}_{\mu\nu} \tag{4}$$

in which the original "bare" master function has been replaced by a renormalized "dressed" master function expressible [3, 4] by

$$\tilde{\Lambda} = \Lambda + \hat{\Lambda}_{\mathrm{g}} \tag{5}$$

in which the divergent part of the self-interaction has been absorbed in an adjustment term of the form

$$\hat{\Lambda}_{\mathrm{g}} = \tfrac{1}{4}\overline{T}^{\mu\nu}\hat{h}_{\mu\nu} \tag{6}$$

For any simple elastic string model of the kind considered here, the master function Λ will depend just on the scalar magnitude that is specifiable [6] as $\chi = -\gamma^{ij}\psi_{,i}\psi_{,j} = -p_{\mu}p^{\mu}$, where the relevant momentum vector is defined by $p^{\mu} = x^{\mu}_{,i}\gamma^{ij}\psi_{,j}$. It can be seen that the surface stress-energy tensor of the "bare" model will be expressible in the form

$$\overline{T}^{\mu\nu} = \Lambda\gamma^{\mu\nu} + 2\,\frac{d\Lambda}{d\chi}\,p^{\mu}p^{\nu} \tag{7}$$

using the notation $\gamma^{\mu\nu} = \gamma^{ij}\bar{x}^{\mu}_{,i}\bar{x}^{\nu}_{,j}$ for the fundamental tensor of the worldsheet, i.e., the background spacetime projection of its internal metric. It can thus be seen from (3) that the self-gravitational action contribution (6) will be expressible simply as

$$\hat{\Lambda}_{\rm g} = 2\mathrm{G}\,\hat{l}\left(\chi\,\frac{d\Lambda}{d\chi}\right)^2 \tag{8}$$

4. PHYSICAL ADMISSIBILITY CONDITIONS

The preceding application [3] of the formula (8) was to a smoothed average description of the effect of short-wavelength wiggles on an underlying Nambu–Goto string as described by the transonic model for which the relevant Lagrangian master function is specified by a constant mass parameter as a function of the special form $\Lambda = -m\sqrt{m^2 - \chi}$, which (unfortunately as far as its convenient integrability properties are concerened) will not be preserved by the renormalization. The purpose of the present paper is to show, however, that there does exist a simple but nontrivial category of models whose algebraic character is preserved by the adjustment given by (8), and that are thus "formally" renormalizable in the sense that all that is required is a rescaling of the constant parameters characterizing the model.

The simplest example of a subcategory that is "formally" renormalizable in this sense is of course the one constituted by the Nambu–Goto models, for which the master function is just a constant, $\Lambda = -m^2$, where m has the dimensions of mass. There has never been any doubt about the "formal" renormalizability of this subcategory, since it was always supposed that the appropriately renormalized model would be given by another constant, $\tilde{\Lambda} = -\widetilde{m}^2$. However, according to the formula that was commonly quoted in textbooks [7] for many years, the "dressed" value was given in terms of the the logarithmic regularization factor $\hat{l}$ specified above by $\widetilde{m}^2 = m^2(1 - 4\mathrm{G}\,m^2\hat{l})$, whereas a more careful calculation [2] has recently shown that the correct value, as obtainable directly from (8), is simply $\widetilde{m}^2 = m^2$. In other words the renormalization in the Nambu–Goto case is trivial in the sense that it has no effect at all.

Having made the observation that the Nambu–Goto category is—from this point of view—trivial, one is left with the question of the existence of a category that would be "formally" renormalizable in a nontrivial manner. As will be shown explicitly below, it is very easy to construct a master function that satisfies the "formal" renormalizability condition, but what is not quite so easy is to ensure that the resulting model also satisfies the further requirements needed for physical admissibility. In order for the energy densty and tension to be positive it is necessary that the (on-shell) value of the master function Λ and of its dual [8] (in the Hodge sense with respect to the two-dimensional geometry of the string world sheet) as given by [6]

$$^*\Lambda = \Lambda - 2\chi \frac{d\Lambda}{d\chi} \tag{9}$$

should both be negative,

$$\Lambda < 0, \qquad ^*\Lambda < 0 \tag{10}$$

and in order to avoid microscopic instability on one hand, and causality violation on the other hand, both the extrinsic ("wiggle"-type) perturbation propagation speed $c_{\rm E}$ and the longitudinal (sound-type) perturbation propagation speed $c_{\rm L}$ must be real and less than unity (assuming units such that the speed of light is itself unity), the range of physical validity of an elastic string model is restricted by the conditions [9]

$$0 < c_{\rm E}^2 \leq 1, \qquad 0 < c_{\rm L}^2 \leq 1 \tag{11}$$

in which the quantities $c_{\rm E}^2$ and $c_{\rm E}^2$ are given by the formulas

$$c_{\rm E}^{\pm 2} = \frac{^*\Lambda}{\Lambda}, \qquad c_{\rm L}^{\pm 2} = -\frac{d^*\Lambda}{d\Lambda} \tag{12}$$

where the sign $\pm$ is defined to be positive, $\pm = +$, wherever the current is timelike, i.e., $\chi < 0$, and negative, $\pm = -$, wherever the current is spacelike, i.e., $\chi > 0$, while in the null limit $\chi = 0$, the requirement reduces to $^*\Lambda = \Lambda$ and $d^*\Lambda/d\Lambda = -1$.

It will be convenient to simplify the foregoing formulas by a change of variable whereby χ is replaced by a new variable $\Xi = \ln\{\chi/\chi_0\}$ for some fixed value χ_0 and to use a dot to denote differentiation with respect to Ξ, so that in particular one has

$$\dot{\Lambda} = d\Lambda/d\Xi = \chi\, d\Lambda/d\chi \tag{13}$$

This enables us to express the dual master function in the form

$$^*\Lambda = \Lambda - 2\,\dot{\Lambda} \tag{14}$$

so that one obtains

$$^*\Lambda/\Lambda = 1 - 2\dot{\Lambda}/\Lambda \tag{15}$$

$$d^*\Lambda/d\Lambda = 1 - 2\ddot{\Lambda}/\dot{\Lambda} \tag{16}$$

The formula for the "dressed" master function will be similarly expressible in the form

$$\tilde{\Lambda} = \Lambda + 2G\hat{l}\dot{\Lambda}^2 \tag{17}$$

This notation can also be used to express the preceding conditions (11) for good physical behavior (causality and local stability) as

$$\Lambda < 2\dot{\Lambda} \leq 2\ddot{\Lambda} < \dot{\Lambda} \leq 0 \tag{18}$$

in the timelike current regime, $\chi > 0$, and as

$$\Lambda < 0 \leq 2\dot{\Lambda} \leq 2\ddot{\Lambda} \tag{19}$$

in the spacelike current regime, $\chi > 0$.

5. "FORMALLY" RENORMALIZABLE MODELS

It is evident fron (17) that, as has already been remarked, the action renormalization $\Lambda \mapsto \tilde{\Lambda}$ will have no effect at all on a master function that is constant (i.e., of Nambu–Goto type). Clearly the simplest category of functions that will be nontrivially preserved by such a transformation consists of those that are linear in Ξ, i.e., those of the form

$$\Lambda = -m^2 + A\Xi \tag{20}$$

where A, like m^2, is a fixed parameter. For a master function of this type one obtains $\dot{\Lambda} = A$ and $\ddot{\Lambda} = 0$, so the effect of the action renormalization will be expressible as a simple parameter renormalization $m^2 \mapsto \widetilde{m^2}$, $A \mapsto \tilde{A}$, of which the latter part is trivial, $\tilde{A} = A$, while the only nontrivial part will be a mass renormalization given by the formula $\widetilde{m^2} = m^2 - 2G\hat{l}A^2$. It can be seen, however, that although it thus satifies the requirement of being renormalizable in the strict "formal" sense, this linear master function does not provide a physically admissible string model, since (unless $\dot{\Lambda}$ also vanishes) the restriction $\ddot{\Lambda} = 0$ is compatible with neither (18) nor (19): as can be seen directly from (16) it is simply inconsistent with the requirement that the "sound" (longitudinal perturbation) speed should be real.

Although the simplest mathematical possibility is thus excluded on physical grounds, we can obtain a "formally renormalizable" category of physically admissible models by going on from linear to quadratic order in Ξ. There is no loss of generality in writing a quadratic function of Ξ in the form

$$\Lambda = -m^2 + C\Xi^2 \tag{21}$$

where C, like m^2, is a fixed parameter. (The reason that, provided the coefficient C is nonzero, no generality can be gained by adding in an extra linear term $A\Xi$ is that such a term could be absorbed into the homogeneously quadratic part by a rescaling of the constant parameter χ_0 that was used to fix the calibration of Ξ.) For a master function of this quadratic type one obtains $\Lambda = 2C\Xi$ and $\Lambda = 2C$. The effect of the action renormalization will therefore be expressible as a simple parameter renormalization $m^2 \mapsto \widetilde{m}^2$, $C \mapsto \tilde{C}$, of which in this case it is the first part that is trivial, $\widetilde{m}^2 = m^2$, while the second part will have the nontrivial form

$$\tilde{C} = C + 8\mathrm{G}\hat{l}C^2 \tag{22}$$

In order for a master function of the form (21) to provide a string model satisfying the physical admissibility conditions recapitulated above, it can be seen to be necessary and sufficient that the parameter m should be nonzero, and that the other parameter C should satisfy the condition

$$3C > -m^2 \tag{23}$$

When C is negative, χ will also have to be negative, i.e., the current is restricted to be timelike, and the permissible range for Ξ will be given by

$$-\frac{m^2}{3} < C < -\frac{m^2}{4} \Rightarrow 1 \leq \Xi < 2\left(1 - \sqrt{1 - \frac{m^2}{4|C|}}\right) \tag{24}$$

(the lower limit being where $c_{\mathrm{L}} \rightarrow 1$ and the upper limit where $c_{\mathrm{E}} \rightarrow 0$) and

$$-\frac{m^2}{4} \leq C < 0 \Rightarrow 1 \leq \Xi < 2 \tag{25}$$

(the lower limit being again where $c_{\mathrm{L}} \rightarrow 1$ and the upper limit where $c_{\mathrm{L}} \rightarrow 0$). When C is positive, χ will also have to be positive, i.e., the current is restricted to be spacelike, and the permissible range for Ξ will be given by

$$0 < C \leq m^2 \Rightarrow 0 \leq \Xi \leq 1 \tag{26}$$

(the lower limit being where $c_{\mathrm{E}} \rightarrow 1$ and the upper limit where $c_{\mathrm{L}} \rightarrow 1$) and

$$C > m^2 \Rightarrow 0 \leq \Xi < \frac{m}{\sqrt{C}} \tag{27}$$

(the lower limit being again where $c_{\mathrm{E}} \rightarrow 1$ and the upper limit where $c_{\mathrm{E}} \rightarrow 0$).

The quadratic formula (21) (for $C > -m^2/3$) provides not only the lowest order function of Ξ that gives a "formally" renormable string model that is physically admissible, but also the highest order function with this property: for example, if we were to include a cubic order term in the "bare" master function, the corresponding "dressed" master function would be not of cubic, but of quartic order.

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